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Mechanical systems, equivalent in Lyapunov's sense to systems not containing non-conservative positional forces☆

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Abstract

Developing results obtained previously (Refs. Koshlyakov VN. Structural transformations of the equations of perturbed motion of a certain class of dynamical systems. *Ukr Mat Zh* 1997; 49 (4): 535–539; Koshlyakov VN. Structural transformations of dynamical systems with gyroscopic forces. *Prikl Mat Mekh* 1997; 61 (5): 774–780; Koshlyakov VN, Makarov VL. The theory of gyroscopic systems with non-conservative forces. *Prikl Mat Mekh* 2001; 65 (4): 698–704; Koshlyakov VN, Makarov VL. The stability of non-conservative systems with degenerate matrices of dissipative forces. *Prikl Mat Mekh* 2004; 68 (6): 906–913), the general problem of eliminating non-conservative positional structures from the second-order differential equation with constant matrix coefficients, obtained when modelling many mechanical systems, is considered. It is assumed that the matrices of the dissipative and non-conservative positional structures may, in particular, be degenerate. Under fairly general assumptions, theorems containing the necessary and sufficient conditions for a Lyapunov transformation to exist are proved. This converts the initial matrix equation to an equivalent autonomous form (in Lyapunov's sense) with a symmetrical matrix of the positional forces. An illustrative example is considered.

1. Formulation of the problem

Initial equations. Consider the equation

$$J\ddot{x}(t) + (D + HG)\dot{x}(t) + (\Pi + P)x(t) = 0 \tag{1.1}$$

with positive scalar parameter H (Ref. [5]5), where $x = \mathrm{col}(x_1(t), \ldots, x_m(t))$ is the required vector and the coefficients J, D, G, Π and P are constant $m \times m$ matrices with real elements, which satisfy the conditions

$$J = J^T > 0, \quad D = D^T \geq 0, \quad G = -G^T, \quad P = -P^T, \quad \Pi = \Pi^T, \quad \det G \neq 0 \tag{1.2}$$

(the superscript T denotes transposition).

No additional limitations, apart from those in conditions (1.2), are imposed on the matrices in Eq. (1.1).

Eq. (1.1) describes the motion of many mechanical systems acted upon by dissipative, gyroscopic and non-conservative positional forces. In systems containing gyroscopes, the parameter H may be understood to be the value of the inherent angular momentum of each of the gyroscopes in the system considered, while J is the matrix of

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the overall moments of inertia of the system about corresponding axes. The elements of the skew-symmetric matrix $G = ||g_{kj}||_{k,j=1,\ldots,m}$ here are determined uniquely directly from the initial equations of the device considered.

Note that in the equations of the perturbed motion of gyroscopic systems, set up on mobile bases, in particular, on a base which rotates with an angular velocity ω about a vertical position, the elements of the matrix Π may contain elements with the factor H, due to the gyroscopic moment, which arises when the base rotates with the angular velocity indicated. Under certain conditions this moment may have a considerable influence on the stability of the system. To estimate this effect, it is convenient to represent the matrix Π first of all in the form of the sum of two matrices, assuming

$$\Pi = \Pi^{(0)} + H\Pi^{(H)} \tag{1.3}$$

where the matrices $\Pi^{(0)}$ and $\Pi^{(H)}$ are independent of H.[5]

In Eq. (1.1) we make the transformation

$$J^{1/2}x(t) = L(t)\xi(t), \quad L(0) = E \tag{1.4}$$

with Lyapunov's matrix $L(t)$, where E is the identity matrix. Transformation (1.4) does not change the stability properties of the trivial solution of Eq. (1.1), corresponding to unperturbed motion. The transformed equation for the vector $\xi(t)$ will have the form

$$\ddot{\xi}(t) + L^{-1}(t)\mathscr{L}^{(1)}(t)\dot{\xi}(t) + L^{-1}(t)\mathscr{L}^{(2)}(t)\xi(t) = 0 \tag{1.5}$$

where

$$\mathscr{L}^{(1)}(t) = 2\dot{L}(t) + (D_1 + HG_1)L(t)$$

$$\mathscr{L}^{(2)}(t) = \ddot{L}(t) + (D_1 + HG_1)\dot{L}(t) + (P_1 + \Pi_1)L(t)$$

The subscript unity on the matrices denotes their multiplication from the left and from the right by the matrix $J^{-1/2}$, for example $\Pi_1 = J^{-1/2}\Pi J^{-1/2}$.

Consider the following problem: it is required to obtain the necessary and sufficient conditions for which, when they are satisfied, transformation (1.4) exists with a Lyapunov matrix $L(t)$, which is independent of H, which converts Eq. (1.1) to the equivalent form (in Lyapunov's sense)

$$\ddot{\xi}(t) + V_1\dot{\xi}(t) + W_1\xi(t) = 0, \quad W_1 = W_1^T \tag{1.6}$$

where V_1 and W_1 are certain constant matrices, which depend on the parameter H.

Eqs. (1.5) and (1.6) will be said to be equivalent if each solution of Eq. (1.5) is a solution of Eq. (1.6), and conversely. Note that, in the formulation of the problem above it is not required that the Lyapunov matrix should be initially determined from the equation

$$D\dot{L}(t) + PL(t) = 0 \tag{1.7}$$

as was assumed earlier in Refs. 1–4,6,7.

We will henceforth use the following notation. For any two $m \times m$ matrices A and B we will denote their commutator by $[A, B] = AB - BA$. If the matrix A is degenerate, we will denote the pseudoinverse Moore-Penrose matrix to the matrix A by A^+ (Refs. 8–10).

2. Theorem 1 on the equivalence of Eqs. (1.1) and (1.6) (in Lyapunov's sense)

We will first prove the following lemma.

Lemma. *Suppose $D_1 = D_1^T \geq 0$, $C = -C^T$, where C is a certain constant skew-symmetric matrix. Then, in order that the matrix*

$$L(2t) = \exp(-D_1 t)\exp((D_1 + C)t), \quad t \geq 0 \tag{2.1}$$

should be a Lyapunov matrix, it is necessary and sufficient that

$$[D_1, C] = 0 \tag{2.2}$$

Proof. Suppose the commutativity condition (2.2) is satisfied. Expression (2.1) then takes the form

$$L(2t) = \exp(Ct), \quad t \geq 0 \tag{2.3}$$

Since we have assumed that the matrix C is skew-symmetric, $L(2t)$ will be a Lyapunov matrix.[8] □

We will prove the inverse assertion. Suppose matrix (2.1) is a Lyapunov matrix. Then

$$\|L(2t)\| \leq K < \infty, \quad \forall t \geq 0 \tag{2.4}$$

where $||L(2t)||$ is the norm of the matrix $L(2t)$ and K is a positive constant. It is necessary to show that the commutativity condition (2.2) will then be satisfied.

Since the matrix D_1 is symmetrical, the following representation[8] holds for it

$$D_1 = S\Lambda S^T, \quad \Lambda = \mathrm{diag}(\lambda_k)_{k=1,\ldots,m} \tag{2.5}$$

where S is the fundamental matrix and Λ is the diagonal matrix of the eigenvalues λ_k of the matrix D_1. We then have the relations

$$\begin{aligned} &SS^T = S^TS = E, \quad \lambda_k \geq 0, \quad k = 1, 2, \ldots, m \\ &L(2t) = S\mathrm{diag}(\exp(-\lambda_k t))_{k=1,\ldots,m}\exp((\Lambda + S^TCS)t)S^T \\ &\mathrm{tr}D_1 = \mathrm{tr}(\Lambda + S^TCS) = \sigma \end{aligned} \tag{2.6}$$

We will denote by μ_k ($k=1, \ldots, m$) the eigenvalues of the matrix

$$N = (n_{ij})_{i,j=1,\ldots,m} = \Lambda + S^TCS$$

We will choose the numbering of the eigenvalues λ_k and μ_k ($k=1, \ldots, m$) such that

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m, \quad \mathrm{Re}\mu_1 \geq \mathrm{Re}\mu_2 \geq \ldots \geq \mathrm{Re}\mu_m$$

We will prove the inverse assertion indirectly. Suppose inequality (2.4) holds, but the commutativity condition (2.2) is not satisfied. Then not all the λ_k will be identical. We will assume, without loss of generality, that

$$\lambda_1 = \lambda_2 = l_1 > \lambda_3 = \lambda_4 = \ldots = \lambda_m = l_2$$

We reduce the matrix N to Jordan form[8]

$$N = TMT^{-1}$$

where $T = (t_{i,j})_{i,j=1,\ldots,m}$ and M is an upper triangular matrix consisting of Jordan boxes, where

$$\mathrm{diag}M = \mathrm{diag}(\mu_k)_{k=1,\ldots,m}$$

It follows from relations (2.6) that

$$L(2t) = S\mathrm{diag}(\exp(-\lambda_k t))_{k=1,\ldots,m}T\exp(Mt)T^{-1}S^T$$

Since S^TCS is a skew-symmetric matrix, $\mu_1 + \mu_2 + \ldots + \mu_m = \sigma$. Further, we have

$$\begin{aligned} \|L(2t)\|^2 &= \left\{\sup_{y\neq 0}\frac{\|L(2t)y\|}{\|y\|}\right\}^2 \geq \kappa^{-1}\|L(2t)STe_1\|^2 = \kappa^{-1}\|S^TL(2t)STe_1\|^2 = \\ &= \kappa^{-1}\|\mathrm{diag}(\exp((-\lambda_k + \mu_1)t))_{k=1,\ldots,m}Te_1\|^2 = \\ &= \kappa^{-1}\|\mathrm{col}(\nu_{11}(t)t_{11}, \nu_{11}(t)t_{21}, \nu_{21}(t)t_{31}, \ldots, \nu_{21}(t)t_{m1})\|^2 \end{aligned}$$

where

$$\kappa = \|STe_1\|^2, \quad e_1 = \mathrm{col}(\delta_{1k})_{k=1,\ldots,m}, \quad \nu_{\alpha 1}(t) = \exp((-l_\alpha + \mu_1)t), \quad \alpha = 1, 2$$

where δ_{ik} is the Kronecker delta.

The following relation holds

$$l_1 \geq \mathrm{Re}\mu_1 > l_2 \tag{2.7}$$

In fact, if $\mathrm{Re}\mu_1 > l_1$, the vector $S^T L(2t)STe_1$ will have a bounded norm for all $t \geq 0$, when the first column of the non-singular matrix T is a null matrix, which is impossible. If $\mathrm{Re}\mu_1 \leq l_2$, we have

$$\sigma = \sum_{k=1}^{m} \lambda_k = 2l_1 + (m-2)l_2 > \sum_{k=1}^{m} \mu_k \tag{2.8}$$

which is also impossible. It follows from relation (2.7) that the vector $S^T L(2t)STe_1$ will have a boundednorm for all $t \geq 0$, when

$$t_{31} = t_{41} = \ldots = t_{m1} = 0$$

The latter denotes that the matrix N will have as its own eigenvector the vector $T_1 = \mathrm{col}(t_{11}, t_{21}, 0, \ldots, 0)$, corresponding to the eigenvalue μ_1. The matrix

$$N_{11} = (n_{ij})_{i,j=1,2}$$

will then have its own eigenvector $\mathrm{col}(t_{11}, t_{21})$, which corresponds to the eigenvalue μ_1. Since $n_{12} = -n_{21}$ and $n_{11} = n_{22} = l_1$, the matrix N_{11} will be a matrix of simple structure with eigenvalues μ_1 and μ_2 and corresponding eigenvectors $t_k = \mathrm{col}(t_{1\mathrm{k}}, t_{2\mathrm{k}})$ $(k = 1, 2)$, which are linearly independent. Hence it follows that the matrix N will also have as its own eigenvector the vector $T_2 = \mathrm{col}(t_{12}, t_{22}, 0, \ldots, 0)$, which corresponds to the eigenvalue μ_2, and the following relations must be satisfied

$$n_{1k}t_{1s} + n_{2k}t_{2s} = 0, \quad s = 1, 2, \quad k = 3, 4, \ldots, m \tag{2.9}$$

Since the vectors $t_s = \mathrm{col}(t_{1s}, t_{2s})$ $(s = 1, 2)$ are linearly independent, from relations (2.9) we obtain

$$n_{1k} = n_{2k} = 0, \quad k = 3, 4, \ldots, m$$

which, taking into account the fact that the matrix $S^T CS$ is skew-symmetric, leads to the commutativity of the matrices D_1 and C, but this contradicts the assumption.

Theorem 1. *Suppose the matrix coefficients in Eq. (1.1) satisfy conditions (1.2)* and *(1.3). In order that the Lyapunov transformation (1.4), which reduces Eq. (1.1) to the form (1.6) with a matrix W_1 that is symmetrical for all $H > 0$, should exist, independent of H, it is necessary and sufficient that the system of equations*

$$\begin{gathered} [G_1, C] = 0, \quad [D_1, C] = 0, \quad G_1C + C^TG_1 = 0 \\ D_1C = -2P_1, \quad [C, \Pi_1^{(H)}] = 0, \quad [C, \Pi_1^{(0)}] = 0 \end{gathered} \tag{2.10}$$

should have as its own solution a matrix C which generates the Lyapunov matrix

$$L(t) = \exp(Ct/2) \tag{2.11}$$

Here

$$\begin{gathered} V_1 = D_1 + HG_1 + C, \quad 4W_1 = C^2 + 2HG_1C + 2D_1C + 4(P_1 + \Pi_1) \\ \Pi_1 = \Pi_1^{(0)} + H\Pi_1^{(H)}, \quad \Pi_1^{(0)} = J^{-1/2}\Pi^{(0)}J^{-1/2}, \quad \Pi_1^{(H)} = J^{-1/2}\Pi^{(H)}J^{-1/2} \end{gathered} \tag{2.12}$$

Proof. Necessity. Suppose a Lyapunov transformation (1.4) exists, independent of H, by means of which Eq. (1.1) is converted to the form (1.6). Then we will have the following identities

$$\mathcal{L}^{(1)}(t) = L(t)V_1, \quad \mathcal{L}^{(2)}(t) = L(t)W_1, \quad \forall t \geq 0 \tag{2.13}$$

Referring to the first identity of (2.13), we will consider it as an equation in $L(t)$. Its solution, which satisfies the condition $L(0)=E$, has the form

$$L(t) = \exp(At)\exp(Bt), \quad 2A = -(D_1 + HG_1), \quad 2B = V_1 \tag{2.14}$$

Substituting solution (2.14) into the second identity of (2.13) and assuming further that $t=0$, we obtain the expression

$$W_1 = B^2 - A^2 + P_1 + \Pi_1 \tag{2.15}$$

taking which into account together with solution (2.14), from the second identity of (2.13) we obtain the commutativity condition

$$[-A^2 + P_1 + \Pi_1, L(t)] = 0, \quad t \geq 0 \tag{2.16}$$

Differentiating the equality obtained with respect to t and assuming $t=0$, taking the notation of (2.14) into account, we obtain the equation

$$[P_1 + \Pi_1, D_1 + HG_1] - [-(D_1 + HG_1)^2/4 + P_1 + \Pi_1, V_1] = 0$$

from which we find the matrix V_1 in the form of the first relation of (2.12), where C is, for the present, an arbitrary matrix, which satisfies the commutativity condition

$$[-(D_1 + HG_1)^2/4 + P_1 + \Pi_1, C] = 0 \tag{2.17}$$

Taking the first relation of (2.12) into account and bearing in mind that the matrix $L(t)$ is independent of H, we arrive from the first identity of (2.13) to the identity

$$[G_1, L(t)] = 0 \quad t \geq 0 \tag{2.18}$$

We differentiate identity (2.18) with respect to t and put $t=0$. Then, taking the first identity of (2.13) and the first relation of (2.12) into account, we obtain that the matrix C satisfies the first equation of (2.10).

Taking the first equation of (2.10) and the first relation of (2.12) into account, we can represent expression (2.15) in the form

$$4W_1 = C^2 + 2HG_1C + (D_1C + CD_1) + 4(P_1 + \Pi_1) \tag{2.19}$$

We will consider the second identity of (2.13) with the matrix W_1, defined by formula (2.19), as an equation in $L(t)$. In order that it should have a solution independent of H, it is necessary that the matrix $L(t)$ should satisfy the equation

$$G_1\dot{L}(t) + \Pi_1^{(H)}L(t) = L(t)G_1C/2 + L(t)\Pi_1^{(H)}, \quad t \geq 0 \tag{2.20}$$

We will now obtain the conditions for which the matrix (2.19) is symmetrical for all $H>0$. This will occur when a relation, identical with the third equation of (2.10), is satisfied as well as the relation

$$(C^2 - (C^T)^2)/4 + (D_1C + CD_1 - C^TD_1 - D_1C^T)/4 + 2P_1 = 0 \tag{2.21}$$

which follows from representation (2.19) when $W_1 = W_1^T$ for all $H>0$.

It follows from the first and third equations of system (2.10) and the condition $\det C \neq 0$, that the matrix C is skew-symmetric. Further, we obtain from the first identity of (2.13), identity (2.18) and the first relation of (2.12) that the Lyapunov matrix $L(t)$, independent of H, should have the form

$$L(2t) = \exp(-D_1t)\exp(D_1 + C)t \tag{2.22}$$

We now use the lemma proved above. It follows from this lemma that the matrix (2.11) will be a Lyapunov matrix when the second equation (2.10), applicable to the matrix C, is satisfied. The matrix $L(t)$ then takes the form (2.11). Taking into account the fact that the matrix C is skew-symmetric and the commutativity of the matrices D_1 and C, we obtain that relation (2.21) takes the form of the fourth equation of system (2.10).

The fifth equation of (2.10) is obtained from representation (2.11) and relations (2.20).

It remains to satisfy the last equation of system (2.10). We differentiate the second identity of (2.13) with respect to t and put $t=0$. Then, using representation (2.11), and also the first, second and fourth equations of (2.10), it can be shown that the matrix C satisfies the sixth equation of (2.10).

Thus, the system of Eq. (2.10) has the matrix C, which generates the Lyapunov matrix (2.11), as its solution. This proves the necessity.

Sufficiency. We will assume that the system of Eq. (2.10) is compatible and we will suppose that the matrix C is the solution of this system (or one of the solutions, if it is not unique). Corresponding to this solution, we construct the matrix (2.11). We conclude from the first and third equations of system (2.10) and, taking the condition $\det G \neq 0$ into account, that the matrix C is skew-symmetric. Consequently, the matrix (2.11) is a Lyapunov matrix independent of H. We will show that the Lyapunov transformation (1.4), (2.11) then reduces Eq. (1.1) to the form (1.6).

Eq. (1.1), after transformations (1.4) and (2.11), takes the form of the equivalent Eq. (1.5). Consider the coefficient of $\dot{\xi}(t)$. To convert it we will use representation (2.11) and the first two equations of system (2.10). We obtain

$$L^{-1}(t)\mathcal{L}^{(1)}(t) = C + D_1 + HG_1 = V_1$$

We similarly consider the coefficient of $\xi(t)$. To convert it we will use representation (2.2), and also the first, second, fifth and sixth equations of (2.10). As a result we obtain

$$L^{-1}(t)\mathcal{L}^{(2)}(t) = C^2/4 + HG_1C/2 + D_1C/2 + P_1 + \Pi_1 = W_1$$

Hence, Eq. (1.5) takes the form

$$\ddot{\xi}(t) + V_1\dot{\xi}(t) + W_1\xi(t) = 0$$

Using the third and fourth equations of (2.10) it can be shown that the matrix W_1 is symmetrical for all $H>0$, i.e. $W_1 = W_1^T$. This proves the sufficiency.

As a consequence of the theorem which has been proved we have from the fourth equation of system (2.10) (Ref. 9)

$$C = -2D_1^+P_1 + (E - D_1^+D_1)Y \tag{2.23}$$

where Y is an arbitrary $m \times m$ matrix, and D_1^+ is the pseudoinverse Moore-Penrose matrix to the matrix D_1. The condition for the fourth equation of system (2.10) to be solvable must then be satisfied, namely,

$$(E - D_1D_1^+)P_1 = 0 \tag{2.24}$$

3. The case $D>0$

The results obtained in Section 2 can be extended to the case when the matrix D of the dissipative forces is positive-definite and, consequently, non-degenerate.

Theorem 2. *Suppose conditions (1.2) are satisfied and*

$$D>0 \tag{3.1}$$

In order that Lyapunov transformation (1.4), independent of H, should exist, by means of which Eq. (1.1) is converted to the form (1.6) with matrix $W_1 = W_1^T$ for all $H>0$, it is necessary and sufficient for the following relations to be satisfied

$$\begin{gathered} GD^{-1}P = PD^{-1}G, \quad DJ^{-1}P = PJ^{-1}D \\ \Pi^{(0)}D^{-1}P = PD^{-1}\Pi^{(0)}, \quad \Pi^{(H)}D^{-1}P = PD^{-1}\Pi^{(H)} \end{gathered} \tag{3.2}$$

The corresponding Lyapunov matrix has the form

$$L(t) = \exp(-D_1^{-1}P_1t) \tag{3.3}$$

Here

$$V_1 = D_1 + HG_1 - 2D_1^{-1}P_1$$
$$W_1 = (D_1^{-1}P_1)^2 - HG_1D_1^{-1}P_1 + \Pi^{(0)} + H\Pi_1^{(H)} \tag{3.4}$$

Proof. We will use system (2.10) taking condition (3.1) into account. From the fourth equation of this system we obtain

$$C = -2D_1^{-1}P_1 = -2J^{1/2}D^{-1}PJ^{-1/2} \tag{3.5}$$

Note that expression (3.5) follows directly from formula (2.23), since, in the case considered, the pseudoinverse matrix D^+ is identical with the matrix D^{-1}.

Further, we have from the first and third equations of system (2.10)

$$C + C^T = 0 = 2J^{-1/2}(PD^{-1}J - JD^{-1}P)J^{-1/2} =$$
$$= 2J^{1/2}(J^{-1}PD^{-1} - D^{-1}PJ^{-1})J^{1/2} = 2J^{1/2}D^{-1}(DJ^{-1}P - PJ^{-1}D)D^{-1}J^{1/2} \tag{3.6}$$

which leads to the second relation of (3.2).

Substituting expression (3.5) into the first equation of system (2.10) and taking the second relation of (3.2) into account, we have

$$[G_1, C] = 0 = 2J^{1/2}(D^{-1}PJ^{-1}G - J^{-1}GD^{-1}P)J^{-1/2} =$$
$$= 2J^{-1/2}(PD^{-1}G - GD^{-1}P)J^{-1/2}$$

whence the first relation of (3.2) follows.

Using the result of substituting expression (3.5) into the fifth equation of (2.10) and taking the second relation of (3.2) into account, we have

$$-\frac{1}{2}[\Pi_1^{(H)}, C] = J^{-1/2}\Pi^{(H)}D^{-1}PJ^{-1/2} - J^{1/2}D^{-1}PJ^{-1}\Pi^{(H)}J^{-1/2} =$$
$$= J^{-1/2}(\Pi^{(H)}D^{-1}P - PD^{-1}\Pi^{(H)})J^{-1/2} = 0$$

Hence we obtain the fourth relation of (3.2). It can similarly be shown that the third relation (3.2) follows from the sixth equation of system (2.10).

Hence, the solvability of system of (2.10) with conditions (1.2) and (3.1) is equivalent to the satisfaction of relations (3.2) and (3.5). This completes the proof of Theorem 2. □

In conclusion, we note that conditions (3.2) differ from the corresponding conditions obtained previously in Ref. 3, by virtue of the difference of the initial formulations of the problems.

4. A four-gyroscope gyrohorizontal

As an example of the above theory we will consider below a model of a corrected four-gyroscope gyrohorizontal with a different control algorithm compared with that considered earlier in Ref. 4.

The system consists of a platform, set in gimbals, which is stabilized horizontally by four similar gyroscopes with vertical axes of their housings. The gyroscopes are connected in pairs by antiparallelograms, which ensure that each pair of gyroscopes rotates in the plane of the platform by the same angles in opposite directions. Each pair of gyroscopes is connected by a spring to the inner frame of the suspension. It is assumed that the centre of mass of the system is situated below its geometrical centre. The system is set up on a base which rotates with a constant angular velocity ω about a fixed upward vertical with origin at the geometrical centre of the platform. We will neglect the effect of the rotation of the Earth, which unimportant in this case.

The system is provided with two radial correction circuits, by means of which, unlike the system considered in Ref. 4, two moments are superimposed: relative to the internal frame of the suspension, proportional to the value of the angle γ of rotation of one of the pairs of gyroscopes with respect to the vertical axes of their housings, and a moment about the axis of the housing of the gyroscope of the given pair, proportional to the value of the angle β of rotation of the internal frame (see Eq. (4.1) below).

The equations of perturbed motion of the system, as it applies to the conditions specified above, have the form

$$\begin{aligned} &J_1\ddot{\alpha} + 2H\dot{\delta} + 2H\omega\gamma + Pl\alpha = 0 \\ &J_2\ddot{\delta} - 2H\dot{\alpha} + 2H\omega\beta + c\delta = 0 \\ &J_2\ddot{\gamma} + b_2\dot{\gamma} + 2H\dot{\beta} + 2H\omega\alpha + c\gamma + s_2\beta = 0 \\ &J_3\ddot{\beta} + b_3\dot{\beta} - 2H\dot{\gamma} + 2H\omega\delta - s_1\gamma + Pl\beta = 0 \end{aligned} \tag{4.1}$$

Here we have used the same notation as in Eq. (3.1) of Ref. 4.

Assuming

$$\alpha = x_1, \quad \delta = x_2, \quad \gamma = x_3, \quad \beta = x_4$$

we arrive at a special case of Eq. (1.1), in which we must assume

$$x = \mathrm{col}(x_1, \ldots, x_4), \quad J = \mathrm{diag}(J_1, J_2, J_2, J_3), \quad HG = H\mathrm{diag}(T^{(2)}, T^{(2)})$$

$$D = \mathrm{diag}(O, D_{22}), \quad D_{22} = \mathrm{diag}(b_2, b_3), \quad P = \|P_{\alpha\beta}\|_{\alpha,\beta=1,2} = s\mathrm{diag}(O, T^{(1)})$$

$$\Pi = \|\Pi_{\alpha\beta}\|_{\alpha,\beta=1,2}, \quad \Pi_{11} = \mathrm{diag}(Pl, c), \quad \Pi_{12} = \Pi_{21} = 2H\omega E_2, \quad \Pi_{22} = \begin{Vmatrix} c & m \\ m & Pl \end{Vmatrix} \tag{4.2}$$

$$O = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}, \quad E_2 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad T^{(k)} = \begin{Vmatrix} 0 & k \\ -k & 0 \end{Vmatrix} \quad s = \frac{s_2 + s_1}{2}, \quad m = \frac{s_2 - s_1}{2}$$

Hence,

$$\Pi^{(0)} = \mathrm{diag}(\Pi_{11}, \Pi_{22}), \quad H\Pi^{(H)} = 2H\omega\begin{Vmatrix} O & E_2 \\ E_2 & O \end{Vmatrix}$$

As stated above in expressions (4.2) we mean by H the natural angular momentum of each of the gyroscopes of the system.

Since the matrix G is non-degenerate, we can use Theorem 1 and Eq. (2.10) corresponding to it. From the second and third equations of system (2.10) we obtain $C = -C^T$, and hence

$$C = \begin{Vmatrix} c_{12}T^{(1)} & C_{12} \\ -C_{12}^T & c_{34}T^{(1)} \end{Vmatrix}, \quad C_{12} = \begin{Vmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{Vmatrix}$$

Further, using the fourth equation of system (2.10), we obtain

$$c_{13} = c_{14} = c_{23} = c_{24} = 0, \quad c_{34} = -\frac{2s}{b_2}\sqrt{\frac{J_2}{J_3}} = -\frac{2s}{b_3}\sqrt{\frac{J_3}{J_2}}$$

which leads to the conditions

$$C_{12} = 0, \quad b_3J_2 = b_2J_3 \tag{4.3}$$

When conditions (4.3) are satisfied, the first three equations of system (2.10) are also satisfied. If, in accordance with the Sommerfeld-Greenhill concept, we take $b_s = \mu J_s$ ($s = 2, 3$), $\mu > 0$ in the second condition of (4.3), it will be satisfied automatically.

The fifth equation of (2.10) reduces, in the final analysis, to the conditions

$$J_1 = J_3, \quad c_{12} = c_{34} \tag{4.4}$$

Using the sixth equation of (2.10), we obtain

$$m = 0, \quad J_1 c = J_2 Pl \tag{4.5}$$

Condition (2.24) for the fourth equation of (2.10) to be solvable in this problem is also satisfied. In fact, taking expressions (4.2) into account, we have

$$(E_4 - D_1 D_1^+)P_1 = J^{-1/2}\mathrm{diag}(E_2, O)s\,\mathrm{diag}(O, T^{(1)})J^{-1/2}$$

whence it follows that all the elements of the matrix on the left-hand side of this equality are zeros.

Here we must assume that

$$D_1^+ = J^{1/2}D^+J^{1/2}, \quad P_1 = J^{-1/2}PJ^{-1/2}, \quad D^+ = \mathrm{diag}(O, D_{22}^{-1})$$

Hence, in the problem considered here, when the matrix D is degenerate, it turns out that

$$D_1 D_1^+ \neq E_4 \tag{4.6}$$

and hence conditions (3.2), obtained as they apply to the non-degenerate matrix D, in this case turn out to be inapplicable. Hence it follows that under these conditions one cannot formally replace D^{-1} by D^+, as was done by Müller.[7]

Hence, the conditions

$$s_1 = s_2, \quad b_3 J_2 = b_2 J_3, \quad cJ_1 = PlJ_2, \quad J_1 = J_3 \tag{4.7}$$

in which we must assume that $c > 0$, $Pl > 0$, $s_1 > 0$, are necessary and sufficient for the initial system (4.1) with degenerate matrices D and P to be reducible to the form (1.6), not containing non-conservative positional structures. In this case, the matrix C and the Lyapunov matrix $L(t)$ generated by it have the form

$$C = -\frac{2s}{b_2}\sqrt{\frac{J_2}{J_3}}\mathrm{diag}(T^{(1)}, T^{(1)}), \quad L(2t) = \exp(Ct) \tag{4.8}$$

The sufficient criterion for the system considered to be stable can be obtained from the conditions for the symmetric matrix W_1, defined by the second of representations (2.12), to be positive-definite. Taking conditions (4.7) into account we will assume that $J = (J_3, J_2, J_2, J_3)$.

Using the fourth equation of (2.10), we obtain the following expression for the matrix W_1

$$W_1 = \frac{1}{4}(C^2 + 2HG_1C) + \Pi_1$$

whence, taking conditions (4.7) into account, it follows that

$$W_1 = \begin{Vmatrix} w_{11}E_2 & w_{13}E_2 \\ w_{13}E_2 & w_{11}E_2 \end{Vmatrix} \tag{4.9}$$

where

$$w_{11} = -\frac{s^2 J_2}{b_2^2 J_3} + \frac{2Hs}{b_2 J_3} + \frac{Pl}{J_3}, \quad w_{13} = \frac{2H\omega}{\sqrt{J_2 J_3}} \tag{4.10}$$

Sylvester's criterion, applied to matrix (4.9), reduces to the inequalities

$$w_{11} > 0, \quad w_{11}^2 - w_{13}^0 > 0 \tag{4.11}$$

since in this case $\det W_1 > 0$.

The first condition of (4.11) can be represented in the form

$$s(2Hb_2 - sJ_2) + Plb_2^2 > 0 \tag{4.12}$$

Satisfaction of condition (4.12) ensures that nutational oscillations are suppressed.

The second condition of (4.11) reduces to the form

$$s(2Hb_2 - sJ_2) + b_2^2\left(Pl - 2H\omega\sqrt{\frac{J_3}{J_2}}\right) > 0 \tag{4.13}$$

Like the system considered previously in Ref. 4, by virtue of the degeneracy of the matrix D, Rayleigh's function $2\Phi = b_2\dot{\gamma}^2 + \dot{\beta}^2$ will not be positive-definite for all velocities $\dot{\alpha}$, $\dot{\delta}$, $\dot{\gamma}$, $\dot{\beta}$. However, the dissipation in Eq. (4.1) may turn out to be complete. We will denote the determinant of the matrix of the positional forces $\Pi + P$ by Δ. We will then have[4]

$$\Delta = (4H^2\omega^2 - cPl)^2 + cPls_1s_2 \tag{4.14}$$

Since the determinant (4.14) in this problem is non-zero, the dissipative forces acting on system (4.1) vanish only in unperturbed motion, corresponding to the equilibrium position

$$\alpha = \delta = \gamma = \beta = 0$$

This indicates that the dissipation is complete.[11] Then, when the matrix W_1 is positive-definite, to which satisfaction of conditions (4.12) and (4.13) corresponds in this case, the presence of forces with complete dissipation in the gyroscopic forces ensures the property of asymptotic stability in this system.[12]

In the precessional formulation we must retain in condition (4.13) only terms containing the parameter H with a factor. The corresponding condition has the form[4,13]

$$s^2 > \omega^2 b_2 b_3$$

In conclusion, we note that conditions (4.7) are more general compared with the conditions obtained in Ref. 4, as they apply to a system close in structure to the one considered. This is due to the fact that, in this paper, we have not used the second condition of (1.5) in Ref. 4, which impose additional limitations on the system parameters.

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